TECHNIQUES OF INTEGRATION

We have already considered various methods of finding derivatives of given functions. Now we shall take up the converse problem: Given a function $f(x)$, it is required to find a funcuion $F(x)$ such that its derivatives equals $f(x)$, that is to find $F(x)$ such that

$$\frac{d}{dx}[F(x)] = f(x)$$

or $$F'(x) = f(x).$$

Antiderivatives

(4.1) Definition. Let $f(x)$ be a given function. If there is a differentiable function $F(x)$ such that $\frac{d}{dx}[F(x)] = f(x)$, then $F(x)$ is called an **antiderivative** (or **primitive**) of $f(x)$. It may be noted that if $F(x)$ is an antiderivative of $f(x)$ then so is $F(x) + c$, where c is an arbitrary constant.

(4.2) Definition. If $F(x)$ is an antiderivative of $f(x)$, then the expression $F(x) + c$ is called **indefinite integral** of $f(x)$ and is denoted by

$$\int f(x)\,dx = F(x) + c$$

$\int$ is the **integral sign** and $f(x)$ is called the **integrand.** x is called the **variable of integration** and c is an arbitrary **constant of integration.** The arbitrary constant of integration is usually omitted in practice.

The process of finding antiderivatives is called **integration**

(4.3) Theorem. If $F(x)$ and $G(x)$ are two antiderivatives of $f(x)$, then

$$F(x) - G(x) = \text{A constant}$$

Proof. By definition of antiderivative, we have

$$F'(x) = f(x)$$

and $G'(x) = f(x)$

Suppose that

$$F(x) - G(x) = H(x), \text{ then}$$

$$\frac{d}{dx}[F(x) - G(x)] = H'(x)$$

i.e., $F'(x) - G'(x) = H'(x)$

or $f(x) - f(x) = 0 = H'(x)$

Thus $H'(x) = 0$ showing that $H(x)$ is a constant.

Hence $F(x) - G(x) =$ A constant.

(4.4) Theorem. $\frac{d}{dx}\int f(x)\,dx = f(x)$.

Proof. Suppose $\int f(x)\,dx = F(x)$, then

$$\frac{d}{dx}\int f(x)\,dx = \frac{d}{dx}[F(x)] = F'(x)$$

But $F'(x) = f(x)$, by definition

Hence $\frac{d}{dx}\int f(x)\,dx = f(x)$.

(4.5) Theorem. If a is a constant, then

$$\int a f(x)\,dx = a\int f(x)\,dx.$$

Proof. Differentiating right-hand side of the above equation, we have

$$\frac{d}{dx}\left[a\int f(x)\,dx\right] = a\frac{d}{dx}\int f(x)\,dx$$

$$= a f(x), \text{ by } \mathbf{(4.4)}$$

Thus, $a\int f(x)\,dx$ is antiderivative of $a f(x)$

i.e., $\int a f(x)\,dx = a\int f(x)\,dx$.

(4.6) Theorem. The **antiderivative** of algebraic sum (difference) of two (or more) functions is equal to the sum (difference) of their antiderivatives.

i.e., $\int [f(x) \pm g(x)]\,dx = \int f(x)\,dx \pm \int g(x)\,dx$

Proof. $\frac{d}{dx}\left[\int f(x)\,dx \pm \int g(x)\,dx\right] = \frac{d}{dx}\int f(x)\,dx + \frac{d}{dx}\int g(x)\,dx$

$= f(x) \pm g(x)$ by **(4.4)**.

Thus, by definition of antiderivative

$$\int [f(x) \pm g(x)]\,dx = \int f(x)\,dx \pm \int g(x)\,dx.$$

Table of Integrals

1. $\int c\,dx = cx$

2. $\int x^n\,dx = \frac{x^{n+1}}{n+1},\ n \neq -1$

3. $\int \frac{dx}{x} = \ln|x|,\ x \neq 0$

4. $\int e^x\,dx = e^x$

5. $\int a^x\,dx = \frac{a^x}{\ln a},\ a \neq 1,\ a > 0$

6. $\int \sin x\,dx = -\cos x$

7. $\int \cos x\,dx = \sin x$

8. $\int \sec^2 x\,dx = \tan x$

9. $\int \csc^2 x\,dx = -\cot x$

10. $\int \sec x \tan x\,dx = \sec x$

11. $\int \csc x \cot x\,dx = -\csc x$

12. $\int \tan x\,dx = \ln|\sec x|$

13. $\int \cot x\,dx = \ln|\sin x|$

14. $\int \sec x\,dx = \ln|\sec x + \tan x|$

15. $\int \csc x\,dx = \ln|\csc x - \cot x|$

16. $\int \frac{dx}{\sqrt{1-x^2}} = \arcsin x$ or $-\arccos x$

17. $\int \frac{dx}{1+x^2} = \arctan x$ or $-\text{arccot}\, x$

18. $\int \frac{dx}{x\sqrt{x^2-1}} = \text{arcsec}\, x$ or $-\text{arccsc}\, x$

19. $\int \sinh x\,dx = \cosh x$

20. $\int \cosh x\,dx = \sinh x$

21. $\int \text{sech}^2 x\,dx = \tanh x$

22. $\int \text{csch}^2\,dx = -\coth x$

23. $\int \operatorname{sech} x \tanh x \, dx = -\operatorname{sech} x$ 24. $\int \operatorname{csch} x \coth x \, dx = -\coth x$

25. $\int \tanh dx = \ln|\cosh x|$ 26. $\int \coth x \, dx = \ln|\sinh x|$

27. $\int \frac{dx}{\sqrt{1+x^2}} = \sinh^{-1} x = \ln\left(x+\sqrt{x^2+1}\right)$

28. $\int \frac{dx}{\sqrt{x^2-1}} = \cosh^{-1} x = \ln\left(x+\sqrt{x^2-1}\right)$

29. $\int \frac{dx}{1-x^2} = \frac{1}{2}\ln\left|\frac{1+x}{1-x}\right| = \begin{cases}\tanh^{-1} x, & \text{if } |x|<1 \\ \coth^{-1} x, & \text{if } |x|>1\end{cases}$

30. $\int \frac{dx}{x\sqrt{1-x^2}} = \operatorname{sech}^{-1}|x| = -\ln\left(\frac{1+\sqrt{1-x^2}}{x}\right)$

31. $\int \frac{dx}{x\sqrt{1+x^2}} = -\operatorname{csch}^{-1}|x| = -\ln\left(\frac{1}{x}+\frac{\sqrt{1+x^2}}{|x|}\right)$

32. $\int \frac{dx}{x^2-1} = \frac{1}{2}\ln\left|\frac{x-1}{x+1}\right|$

33. $\int \sqrt{a^2-x^2}\, dx = \frac{x\sqrt{a^2-x^2}}{2} + \frac{a^2}{2}\arcsin\left(\frac{x}{a}\right)$

34. $\int \sqrt{x^2+a^2}\, dx = \frac{x\sqrt{x^2+a^2}}{2} + \frac{a^2}{2}\ln\left|\frac{x+\sqrt{x^2+a^2}}{a}\right|$

35. $\int \sqrt{x^2-a^2}\, dx = \frac{x\sqrt{x^2-a^2}}{2} - \frac{a^2}{2}\ln\left|\frac{x+\sqrt{x^2-a^2}}{a}\right|$

Exercise Set 4.1

Write down the indefinite integral of each of the following:

1. 0 2. $\sqrt{x}$ 3. $\frac{1+x}{x}$

4. $\frac{x^2-1}{x^2+1}$ 5. $\tan^2 x$ 6. $\cot^2 x$

7. $\cos^2 x$ 8. $\sin^2 x$ 9. $\sqrt{1-\cos x}$

10. $\sqrt{4-x^2}$ 11. $\sqrt{4+x^2}$ 12. $\sqrt{x^2-4}$